J. DIFFERENTIAL GEOMETRY
27 (1988) 115-136

AN INDEX THEOREM ON OPEN MANIFOLDS. II

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Introduction

This paper is the sequel to [19], in which we proved an abstract index theorem for Dirac-type operators on certain noncompact manifolds. Here we will give some concrete applications of this result, and will also discuss its relationship with L^2 index theorems of Atiyah and Connes.

The set-up for [19] is as follows. Let M be a noncompact oriented Riemannian manifold of bounded geometry, and D a Dirac operator of bounded geometry over it. D is equipped with a grading η, and it will be convenient to use the notations D^+ and D^- for the restrictions of D to the $+1$ and -1 eigenspaces of η. Suppose that M admits a regular exhaustion with corresponding fundamental class $\mathscr{m}$ and trace functional τ. Then the main theorem of [19] computes

$$\dim_\tau(\operatorname{Ind} D) = \langle \mathbf{I}(D), \mathscr{m} \rangle;$$

it identifies a "real-valued index" of D with a "topological" invariant. The fundamental question studied here is: How does the number $\dim_\tau(\operatorname{Ind} D)$ relate to the kernel of D?

Recall from [19, 8.1] the equation

$$\dim_\tau(\operatorname{Ind} D) = \tau(\phi(D^-D^+)) - \tau(\phi(D^+D^-))$$

where ϕ is any Schwartz-class function on $\mathbf{R}^+$ with $\phi(0) = 1$. If the manifold M were compact, one could argue as follows: D has discrete spectrum, hence there is a smooth ϕ of compact support such that $\phi(0) = 1$ and $\phi(\lambda) = 0$ for all nonzero eigenvalues λ of D^2. Then $\phi(D^-D^+)$ is the projection P^+ onto the kernel of D^+, and similarly $\phi(D^+D^-)$ is the corresponding projection P^-, and one gets

$$\dim_\tau(\operatorname{Ind} D) = \tau(P^+) - \tau(P^-), \tag{0.1}$$

Received November 4, 1985 and in revised form, November 25, 1986.

thus obtaining information about the kernels of D^+ and D^-. In the noncompact case this argument does not work, and examples to be presented in §4 show that (0.1) does not hold in general; this paper examines sufficient conditions for (0.1).

To derive such conditions one can pick a standard collection of functions ϕ, namely the heat kernels $\phi_t(\lambda) = e^{-t\lambda}$. As $t \to \infty$, $\phi_t(D^+D^-) \to P^+$ in the strong operator topology on L^2. In fact the corresponding Schwartz kernels converge uniformly on compact subsets of $M \times M$. If the convergence were uniform over all of the diagonal in $M \times M$, then (0.1) would hold, since the fundamental class $\mathscr{m}$ is continuous against the uniform topology. We therefore need estimates on the heat kernel allowing this uniform convergence to be established in certain circumstances.

The contents of this paper are as follows. In §1 we discuss general techniques for obtaining uniform convergence of the heat kernels. These are based on the Bochner method. In §§2, 3, and 4 we apply these techniques to the three classical examples (de Rham operator, Dirac operator, Dolbeault operator). Among the results obtained are restrictions on the scalar and Ricci curvatures of metrics in the quasi-isometry class on noncompact coverings, and a version of the Riemann-Roch theorem for the plane, which was discussed in the introduction to [19]. In §5 we compare our theorem with the results of Atiyah and Connes, and §6 contains further remarks and open questions.

1. Convergence properties of heat kernels

Let S be a Clifford bundle of bounded geometry over the noncompact connected oriented Riemannian manifold M, and let D be the corresponding Dirac operator. Let P denote the orthogonal projection (in $L^2(S)$) onto the kernel of D. This section studies the convergence to P of the heat operators e^{-tD^2} as $t \to \infty$.

Several different Banach spaces of sections of S will play a role, and the reader may care to be reminded of the notation for the various norms introduced in Part I, §2:

$\|\cdot\|$ is the L^2 norm;

$\|\cdot\|_k$ is the norm in the Sobolev space $W^k(S)$;

$\|\|\cdot\|\|_r$ is the norm in the uniform C^r space $UC^r(S)$.

We will also adopt the notational convention that "$f \prec g$" means "f is less than a constant multiple of g," the constant being understood to depend only on the geometry of M and S.

Basic estimates on the heat semigroup are summarized in the next lemma.

(1.1) Lemma. *Let $s \in L^2(S)$, and let $s_t = e^{-tD^2}s$. Then for $t \geqslant 1$,*

(i) *for fixed k and l*, $\|D^l s_t\|_k \prec t^{-l/2}\|s\|$;

(ii) *for fixed r and l*, $\| \|D^l s_t\| \|_r \prec t^{-l/2}\|s\|$.

Proof. (i) follows immediately from the spectral theorem and elementary estimates, and (ii) is a consequence of (i) and the Sobolev embedding theorem.

It follows from the spectral theorem that as $t \to \infty$, $e^{-tD^2} \to P$ in the strong operator topology on $L^2(S)$. In fact, more is true:

(1.2) Lemma. *As $t \to \infty$, the Schwartz kernel of e^{-tD^2} tends to the Schwartz kernel of P in the Fréchet topology of $C^\infty(S \boxtimes S)$ (that is the topology of uniform convergence of all derivatives on compact subsets of $M \times M$). Consequently, the Schwartz kernel of P is uniformly bounded, together with all its derivatives.*

Proof. Since $e^{-tD^2} \to P$ in the strong operator topology on L^2, a fortiori the Schwartz kernel of e^{-tD^2} tends to the Schwartz kernel of P in the weak topology of distributional sections of $S \boxtimes S$. On the other hand, the functions $\lambda \to e^{-t\lambda^2}$ for $t > 1$ form a bounded subset of the space $\mathscr{R}(\mathbf{R})$ of functions of rapid decay on $\mathbf{R}$ [19, 2.13]; hence the Schwartz kernels of the operators e^{-tD^2} form a bounded subset of $UC^\infty(S \boxtimes S)$, hence a fortiori a bounded subset of the Montel space $C^\infty(S \boxtimes S)$. Thus for any sequence $t_j \to \infty$ there is a subsequence of the sequence of the Schwartz kernels of the $e^{-t_jD^2}$ that converges to the Schwartz kernel of P in the topology of $C^\infty(S \boxtimes S)$. Since this topology is metrizable, the result follows.

Now suppose further that M is equipped with a fundamental class m coming from a regular exhaustion, with corresponding trace τ. Unfortunately the convergence of the kernels given by Lemma 1.2 is not strong enough to ensure that $\tau(\eta e^{-tD^2}) \to \tau(\eta P)$ as $t \to \infty$. Uniform convergence of the kernels implies this; but it is convenient for some applications to notice that a slightly weaker concept is also sufficient.

(1.3) Definition. A subset L of M is said to have *density* 0 if for all bounded n-forms α supported within L, $\langle m, \alpha \rangle = 0$. It has *density* 1 if its complement has density 0.

(1.4) Definition. The *full topology* on $UC^0(S)$ is the non-Hausdorff topology defined by the single seminorm

$$\operatorname{Inf}\{\operatorname{Sup}\{|s(x)|: x \in L\}: L \text{ of density } 1\}.$$

Thus the full topology is a kind of "L^∞ topology" relative to m. For brevity we will say that e^{-tD^2} *converges fully* to P if the restriction to the diagonal in $M \times M$ of the Schwartz kernel of e^{-tD^2} converges to the restriction of the

Schwartz kernel of P in the full topology on $UC^0(S \boxtimes S)$. It is easy to see that if e^{-tD^2} converges fully to P, then $\dim_\tau(\mathrm{Ind}\, D) = \tau(\eta e^{-tD^2}) = \tau(\eta P)$, so (0.1) holds.

(1.5) Lemma. *Suppose that for each $\delta > 0$ there is a subset L of M of density 1 such that for all sufficiently large t,*

$$\sup\left\{ \left| e^{-tD^2}s(x) - Ps(x) \right| : x \in L \right\} < \delta \|s\|$$

for all $s \in L^2(S)$. Then e^{-tD^2} converges fully to P.

Proof. Let $\varepsilon_{x,v}$ ($v \in S_x$) denote the distributional section of S defined by

$$\varepsilon_{x,v}(s) = \langle s_x, v \rangle.$$

It follows from the Sobolev embedding theorem [19, 2.8] that there is an integer $k > 0$ for which $\varepsilon_{x,v}$ belongs to $W^{-k}(S)$, with norm bounded independent of x and of the unit vector v. Let Q denote the operator e^{-D^2}. By the spectral theorem, Q maps $W^{-k}(S)$ to $L^2(S)$. Moreover $PQ = P$ and $e^{-tD^2}Q = e^{-(t+1)D^2}$. Thus, by assumption,

$$\left| e^{-(t+1)D^2}\varepsilon_{x,v}(x) - P\varepsilon_{x,v}(x) \right| \prec \delta$$

for $x \in L$ and sufficiently large t, and the result follows.

Estimates of the kind required by (1.5) will be obtained by using the L^2 norm of a section s together with the uniform norm of ∇s to control the uniform norm of s. In making this precise it is helpful to introduce the following concept:

(1.6) Definition. Let c: $(\mathbf{R}^+)^k \to \mathbf{R}^+$ be a function. It will be called an *estimator* if:

(i) c is positively homogeneous of degree 1—that is, for all λ in $\mathbf{R}^+$,

$$c(\lambda x_1, \cdots, \lambda x_k) = \lambda c(x_1, \cdots, x_k);$$

(ii) c is monotone increasing in each variable separately;

(iii) if some x_i tends to zero, the others remaining fixed, then $c(x_1, \cdots, x_k) \to 0$.

For example, $(x_1 \cdots x_k)^{1/k}$ is an estimator. An estimator of estimators is an estimator.

(1.7) Lemma. *Let V: $\mathbf{R}^+ \to \mathbf{R}^+$ be a monotone increasing function such that $V(r) > 0$ for $r > 0$, $V(0) = 0$, and $V(r) \to \infty$ as $r \to \infty$. Then there is an estimator c on $(\mathbf{R}^+)^2$ with the property that*

$$(r/a)V(r/b) \leqslant 1 \Leftrightarrow r \leqslant c(a,b).$$

Proof. Elementary.

The author is indebted to Hörmander for suggesting the idea of the next proposition.

(1.8) Proposition. *Let S be a vector bundle over the (noncompact, connected, oriented) Riemannian manifold M of bounded geometry, equipped with a metric and compatible connection. There is an estimator c: $(\mathbf{R}^+)^3 \to \mathbf{R}^+$ which has the property: for any C^1 section s of S and any subset F of M, if*

(i) $(\int_F |s(x)|^2 \operatorname{vol}(x))^{1/2} \leqslant A_1 > 0$;

(ii) $\sup\{|s(x)|: x \in F\} \leqslant A_2 > 0$;

(iii) $\sup\{|\nabla s(x)|: x \in F\} \leqslant A_3 > 0$;

then there exists $r > 0$ such that

$$|s(y)| \leqslant c(A_1, A_2, A_3) \quad \forall y \in \operatorname{Pen}^-(F, r).$$

Proof. As was shown in Part I (Proposition 5.1), there is a monotone increasing function V_0: $\mathbf{R}^+ \to \mathbf{R}^+$ such that $V_0(0) = 0$, $V_0(r) > 0$ for $r > 0$, $V_0(r) \to \infty$ as $r \to \infty$, and

$$\operatorname{Vol} B(y, r) \geqslant V_0(r) \quad \forall y \in M.$$

Let c_0 be the estimator that corresponds to this function V_0 under the construction of Lemma 1.7, and let c be the estimator

$$c(x_1, x_2, x_3) = \sqrt{c_0(2x_1^2, 4x_2x_3)}\,.$$

Suppose that a section s of S is given which satisfies the conditions (i)–(iii) above. Then write $c = c(A_1, A_2, A_3)$ and $r = c^2/2A_2A_3$.

Let $y \in \operatorname{Pen}^-(F, r)$, and suppose that $|s(y)| \geqslant b$. There is no loss of generality in taking $b < c\sqrt{2}$. Then the ball $B(y, b^2/4A_2A_3)$ is contained within F.

Over this ball, therefore,

$$\left|\nabla |s|^2\right| \leqslant 2|\langle s, \nabla s\rangle| \leqslant 2A_2A_3.$$

Since $|s(y)|^2 \geqslant b^2$, it follows from the mean-value theorem that for all $x \in B(y, b^2/4A_2A_3)$,

$$|s(x)|^2 \geqslant b^2 - 2A_2A_3 \cdot b^2/4A_2A_3 = b^2/2.$$

Thus the integral of $|s|^2$ over this ball is at least $(b^2/2)V_0(b^2/4A_2A_3)$. This integral, however, is at most equal to A_1^2. Therefore

$$(b^2/2A_1^2)V_0(b^2/4A_2A_3) \leqslant 1.$$

By (1.7), then,

$$b^2 \leqslant c_0(2A_1^2, 4A_2A_3) = c^2.$$

Thus it has been shown that if $|s(y)| \geqslant b$, $b < c\sqrt{2}$, then $b \leqslant c$. It follows that $|s(y)| \leqslant c$, as asserted.

Recall now the Weitzenbock formula

$$D^2 = \nabla^*\nabla + R$$

where R is an endomorphism of S constructed from the curvature. For a compact manifold the classical Bochner method argues that D has no kernel if $R > 0$, and that the kernel is strongly restricted if $R = 0$. It is not hard to extend these results to L^2 theory on complete manifolds [9]. Here the Bochner method will be used to obtain full convergence of e^{-tD^2}.

(1.9) Lemma. *The distributional extension of the operator* ∇: $C_c^\infty(S) \to C_c^\infty(S \otimes T^*M)$ *maps* $W^1(S)$ *continuously to* $L^2(S \otimes T^*M)$.

Proof. From the Weitzenbock formula

$$\|\nabla s\|^2 = \|Ds\|^2 - \langle Rs, s\rangle,$$

valid when s is smooth and compactly supported, it follows that ∇ is a continuous linear operator from the dense subspace $C_c^\infty(S)$ of $W^1(S)$ to $L^2(S \otimes T^*M)$. It therefore extends uniquely to a continuous map on $W^1(S)$, which must coincide with the distributional extension of ∇.

(1.10) Definition. A subset L of M will be called *small* if for all $r > 0$ the set $\mathrm{Pen}^+(L, r)$ has density 0. It will be called *large* if its complement is small.

For example, a compact subset of M is small; but there are also noncompact small subsets, such as the x-axis in $\mathbf{R}^2$.

(1.11) Proposition. *Suppose that the restriction of the curvature operator* R *appearing in the Weitzenbock formula to some large subset* F *of* M *is a nonnegative operator. Then* e^{-tD^2} *converges fully to* 0.

Proof. Let L be the complement of F. Given $\delta > 0$, choose [19, 6.3] a smooth function ϕ with values in $[0, 1]$, equal to one on L, equal to zero outside some $\mathrm{Pen}^+(L, r)$, and such that $|\nabla\phi| < \delta$. Let $\psi = 1 - \phi$.

Let $s \in L^2(S)$, and let $s_t = e^{-tD^2}s$. Then

$$\|D^2 s_t\| \, \|s_t\| \geqslant \int (D^2 s_t, s_t)\psi \quad \text{(Cauchy-Schwartz)}$$

$$\geqslant \int (\nabla^*\nabla s_t, s_t)\psi$$

as follows from the Weitzenbock formula, since R is nonnegative on F. We should like to integrate this last expression by parts, to obtain

$$\|D^2 s_t\| \, \|s_t\| \geqslant \int |\nabla s_t|^2 \psi + \int (\nabla s_t, \nabla\psi \otimes s_t).$$

Strictly speaking, the integration by parts is valid only if s_t is compactly supported. However, it follows from (1.9) that both sides of this inequality are continuous functionals on $W^2(S)$. Since s_t may be approximated in the topology of $W^2(S)$ by compactly supported functions, the inequality holds in general.

As $t \to \infty$ we know from (1.1) that $\|D^2 s_t\| \prec t^{-1}\|s\|$, $\|s_t\|_1 \prec \|s\|$. From (1.9) it follows that $\|\nabla s_t\| \prec \|s\|$. By Cauchy-Schwarz, therefore,

$$\left|\int_F (\nabla s_t, \nabla\psi \otimes s_t)\right| \prec \delta\|s\|^2.$$

Therefore, for sufficiently large t,

$$\int_{\mathrm{Pen}^-(F,r)} |\nabla s_t|^2 \leqslant \int |\nabla s_t|^2 \psi \prec \delta\|s\|^2.$$

Now $\|\,\|s_t\|\,\|_2 \prec \|s\|$ as $t \to \infty$, again by (1.1). Therefore (1.8) may be applied to the section ∇s_t of the bundle $S \otimes T^*M$ to show that there is an $r' > r$ such that

$$|\nabla s_t(y)| \prec c(\delta^{1/2}, 1, 1)\|s\| \quad \forall y \in \mathrm{Pen}^-(F, r')$$

for all sufficiently large t. Then (1.8) may be applied again, this time to the section s_t of S, to show that there is an $r'' > r'$ such that

$$|s_t(y)| \prec c(1, 1, c(\delta^{1/2}, 1, 1))\|s\| \quad \forall y \in \mathrm{Pen}^-(F, r'')$$

for sufficiently large t. By (1.5), the desired result follows.

The principle of convergence transfer. So far we have applied the Bochner method only in order to prove the vanishing of the index of the operator D. Some classical applications, such as Lichnerowicz' theorem [15], use the method in this form. In other applications, however, the method is applied to only one of the two terms making up the index. For example, one combines the Riemann-Roch theorem with the Kodaira vanishing theorem in order to show that "sufficiently positive" line bundles have holomorphic sections.

In our general context the curvature operator R appearing in the Weitzenbock formula may be decomposed into the sum of R^+ and R^-, the restrictions of R to the $+1$ and -1 eigenspaces of the grading η. It is evident that if for example $R^+ \geqslant 0$ on some large set, then $e^{-tD^-D^+} \to 0$ fully as $t \to \infty$; the proof is just the same as that of (1.11). We state a consequence of this more formally:

(1.12) Proposition. *If* $R^+ \geqslant 0$ *on a large set, then* $\dim_\tau(\mathrm{Ind}\, D) \leqslant 0$. *If* $R^- \geqslant 0$ *on a large set, then* $\dim_\tau(\mathrm{Ind}\, D) \geqslant 0$.

However, this of course tells us nothing about the convergence as $t \to \infty$ of the other term in the index formula ($e^{-tD^+D^-}$ in the example with $R^+ \geqslant 0$). It turns out that by imposing a stronger positivity condition on R^+ one can obtain full convergence of both terms, and thus get information about the kernel of D^-. In other words the convergence is "transferred" from one term to the other.

(1.13) Proposition. *Suppose that there is a constant k such that $\langle D^2 s, s\rangle \geqslant k\|s\|^2$ for all s in the $+1$-eigenspace of η. (In particular this will be the case if the curvature operator R^+ is uniformly positive.) Then e^{-tD^2} converges fully to P. In particular, formula* (0.1) *of the introduction holds.*

Proof. From (1.12), e^{-tD^2} converges fully to 0 on the $(+1)$-eigenspace of η. By hypothesis, the spectrum of D^-D^+ is contained in the interval $[k, \infty)$. But the spectra of D^-D^+ and D^+D^- are the same apart from 0. The spectral theorem therefore shows that the L^2 operator norm of $e^{-tD^+D^-} - P^-$ is at most e^{-kt}, and by the Sobolev estimates its Schwartz kernel must therefore tend uniformly to zero.

(1.14) Remark. This is not the strongest possible form of the principle of convergence transfer; as in (1.11), we can weaken the condition of uniform positivity to one involving positivity on large subsets. The version given here, however, has a much simpler proof and is adequate for the applications in this paper.

2. The de Rham operator

If one applies the ordinary index theorem to the de Rham operator $d + d^*$, one obtains the Chern-Gauss-Bonnet formula, which equates the integral over a compact manifold of the "Euler form" with the alternating sum of the Betti numbers. On a noncompact manifold of bounded geometry, equipped with a regular exhaustion, we will in this section give two possible definitions of "L^2 Betti numbers." One is more natural, the other is arranged so that an analogue of the Gauss-Bonnet theorem holds; and the convergence problem discussed in §1 reduces to the question: Do these definitions agree?

Let P_k denote the orthogonal projection operator from square-integrable k-forms to square-integrable harmonic k-forms. It is not necessarily a uniform operator; however, it has by (1.2) a uniformly bounded Schwartz kernel, and so the trace $\tau(P_k)$ can be defined. Let $\beta_k = \tau(P_k)$.

We define also

$$\beta_k' = \operatorname{Inf}\{\tau(\phi(\Delta_k)) \colon \phi \in C_c^\infty(\mathbf{R}),\ \phi \geqslant 0,\ \phi(0) = 1\}.$$

Here Δ_k denotes the Laplacian on k-forms. β_k' does not depend (as β_k does) only on the kernel of Δ_k; but it does depend only on the germ near 0 of the spectrum of Δ_k.

These "Betti numbers" have the following properties:

(2.1) $0 \leqslant \beta_k \leqslant \beta_k'$;

(2.2) If $e^{-t\Delta_k}$ converges fully, then $\beta_k = \beta_k'$;

(2.3) (Poincaré duality) If $n = \dim M$, then $\beta_k = \beta_{n-k}$ and $\beta_k' = \beta_{n-k}'$.

Indeed, (2.1) is immediate from the monotoneity property, that $\phi \geqslant 0$ implies $\tau(\phi(\Delta)) \geqslant 0$; (2.2) follows from the definition (1.4) of full convergence; and (2.3) holds since the Hodge $*$-operator intertwines Δ_k and Δ_{n-k}.

The author is not aware of any example with $\beta_k \neq \beta'_k$, so that one could conjecture that the two kinds of Betti numbers are in fact equal. An example to be presented in §4, however, shows that the analogous statement for a different elliptic operator (twisted $\bar\partial$ operator) is definitely false. So any proof would have to depend on special properties of the de Rham operator. Some further remarks on this question may be found in §6.

The index theorem now takes the following form:

(2.4) Theorem (L^2 *Gauss-Bonnet*). *Let $e(TM)$ denote the Euler form of TM. Then*

$$\sum(-1)^k\beta'_k = \langle m, e(TM)\rangle.$$

Proof. For each k there is a sequence ϕ_j^k of nonnegative compactly supported smooth functions equal to 1 at zero such that $\tau(\phi_j^k(\Delta_k)) \to \beta'_k$. For each j let ϕ_j be a function of this sort that is less than or equal to $\min\{\phi_j^k\colon k = 1, \cdots, n\}$. Then by monotoneity

$$\beta'_k \leqslant \tau\big(\phi_j(\Delta_k)\big) \leqslant \tau\big(\phi_j^k(\Delta_k)\big) \to \beta'_k$$

for each k, and so $\tau(\phi_j(\Delta_k)) \to \beta'_k$. On the other hand, $\Sigma(-1)^k\tau(\phi_j(\Delta_k)) = \tau(\eta\phi_j(D^2))$ where D is the de Rham operator, and so, by the main result of Part I, $\Sigma(-1)^k\tau(\phi_j(\Delta_k)) = \langle m, e(TM)\rangle$. The result follows.

(2.5) Proposition. *For a connected and noncompact manifold M, $\beta_0 = \beta'_0 = 0$.*

Proof. We apply the Bochner method via Proposition 1.11. The Weitzenbock formula for zero-forms (i.e. functions) simply reads $D^2f = \nabla^*\nabla f$; the curvature term R is identically zero. (1.11) therefore yields the result.

This has an interesting consequence for surfaces.

(2.6) Proposition. *Let S be a regularly exhaustible, orientable, Riemannian surface of bounded geometry. If the average Gaussian curvature of S (measured with respect to the regular exhaustion) is positive, then S is closed (and hence a sphere).*

Indeed, the average Gaussian curvature is $2\pi(\beta'_0 - \beta'_1 + \beta'_2)$, and $\beta'_0 = \beta'_2$ by Poincaré duality, so the result follows from (2.5).

This observation was previously made by Connes [6] in the context of his index theorem for foliations.

The application to harmonic 1-forms is the original one of the Bochner method. The R term is in this case just the Ricci curvature [4], [11]. Thus from (1.11) one gets

(2.7) Proposition. *If M has nonnegative Ricci curvature on a large subset, then $\beta_1 = \beta'_1 = 0$.*

Notice that if Ric $\geqslant 0$ everywhere, then M has subexponential growth and so automatically possesses a regular exhaustion. The principle of convergence transfer is of little value in this context, since if Ric is uniformly positive then one easily checks using Myers' theorem that M is in fact compact. One example of a noncompact manifold to which (2.7) applies is furnished by a paraboloid of revolution.

Similarly

(2.8) Proposition. *If M has nonnegative Ricci curvature on a large subset and is conformally flat, then all the β_k and β_k' are zero. Consequently $\langle m, e(TM)\rangle = 0$.*

The proof uses (1.11) and (2.4) together with some standard algebra, which may be found in §3.9 of [11].

An important question in geometry asks to what extent the curvature of a manifold may be prescribed. Index theory sometimes yields topological restrictions on prescribing curvatures of compact manifolds, as in Lichnerowicz' theorem. For noncompact manifolds, it is appropriate to employ a finer classification than the topological, as is already shown by the "type problem" of distinguishing the conformal structures of the plane and the disc (cf. [20], where the idea of quasi-isometry is introduced in this connection). Our L^2 index theory yields restrictions on the curvature of metrics within a given quasi-isometry class. For example:

(2.9) Proposition. *Let X be a compact oriented 2-manifold or 4-manifold with $\chi(X) < 0$, and let M be an infinite amenable covering of X. Then there is no metric of bounded geometry in the strict quasi-isometry class on M whose Ricci curvature is nonnegative on a large subset.*

Proof. By [19, 6.6] the manifold M admits a regular exhaustion with fundamental class m such that $\langle m, e(TM)\rangle = \chi(X) < 0$. (The "largeness" of a subset is measured with respect to this exhaustion.) However, $\langle m, e(TM)\rangle$ is an invariant of strict quasi-isometry. If there were a metric in the strict quasi-isometry class on M with nonnegative Ricci curvature on a full subset, then for this metric $\beta_1' = 0$, and in the 4-dimensional case $\beta_3' = 0$ also by Poincaré duality. Therefore by (2.4) $\langle m, e(TM)\rangle \geqslant 0$, a contradiction.

3. The Dirac operator

Suppose now that our noncompact manifold M is equipped with a Spin structure. Then the classical Dirac operator D is defined, and our index theorem gives:

$$\dim_\tau(\operatorname{Ind} D) = \langle m, \hat{A}(TM)\rangle. \tag{3.1}$$

For the operator D it was observed by Lichnerowicz [15] that the curvature term in the Weitzenbock formula is just one-quarter of the usual scalar curvature. Thus from (3.1) and (1.11) there follows:

(3.2) Proposition. *If M has nonnegative scalar curvature on a large set, then* $\langle m, \hat{A}(TM)\rangle = 0$.

Hence, by an argument analogous to that used in the proof of (2.9):

(3.3) Proposition. *Let X be a compact spin manifold with $\hat{A}(X) \neq 0$, and let M be an infinite amenable covering of X. Then there is no metric in the strict quasi-isometry class on M whose scalar curvature is nonnegative on a large subset.*

(An example of such an X is the connected sum of $S^1 \times S^3$ with a quadric hypersurface in $\mathbf{CP}^3$; take $\Gamma = H_1(X, \mathbf{Z})$.)

Kazdan and Warner proved in [14, Theorem 4.3a] that on a compact manifold any function that is negative somewhere is the scalar curvature of some Riemannian metric. A natural analogue of this statement in the context of noncompact manifolds would be the following: Any C^∞-bounded function that is negative somewhere is the scalar curvature of some metric of bounded geometry in the strict quasi-isometry class. Proposition 3.3 shows that this latter statement is false. The set of points where the curvature is negative can be forced to be noncompact.

4. Hirzebruch-Riemann-Roch theorem

Applying the ordinary index theorem to a twisted $\bar{\partial}$ operator on a compact complex manifold, one obtains Hirzebruch's version of the Riemann-Roch theorem. Now let M be a noncompact Kähler manifold of bounded geometry, equipped with a regular exhaustion and with corresponding functionals m and τ, and let V be a hermitian holomorphic vector-bundle of bounded geometry over M. As in §2 we introduced two sorts of "Betti numbers" for M, so here we may introduce "Hodge numbers" for V. Let $P_{p,q}$ be the orthogonal projection operator from square-integrable V-valued (p, q)-forms to $\bar{\partial}$-harmonic square-integrable V-valued (p,q)-forms, and define $h_{p,q}(V) = \tau(P_{p,q})$. On the other hand, let $\Delta_{p,q}$ denote the $\bar{\partial}$ Laplacian on V-valued (p,q)-forms, and set

$$h'_{p,q} = \operatorname{Inf}\{\tau(\phi(\Delta_{p,q})) : \phi \in C_c^\infty(\mathbf{R}), \phi(0) = 1, \phi \geqslant 0\}.$$

The following properties of these Hodge numbers are proved in the same way as the analogous properties of the Betti numbers in §2.

(4.1) $0 \leqslant h_{p,q} \leqslant h'_{p,q}$;

(4.2) If $e^{-t\Delta_{p,q}}$ converges fully to $P_{p,q}$, then $h_{p,q} = h'_{p,q}$;

(4.3) (Kodaira-Serre duality) Let n be the complex dimension of M. Then $h_{p,q}(V) = h_{n-p,n-q}(V^*)$; $h'_{p,q}(V) = h'_{n-p,n-q}(V^*)$;

(4.4) (Hirzebruch-Riemann-Roch) One has

$$\sum(-1)^q h'_{0,q}(V) = \langle m, \mathrm{ch}(V) \cup td(TM)\rangle.$$

It will now be shown how one can combine the principle of convergence transfer and the Kodaira vanishing theorem with the index theorem to get examples with $h_{0,0} = h'_{0,0} > 0$. In other words, the theorem is used to prove the existence of an infinite-dimensional (in the usual sense of linear algebra) space of L^2 holomorphic sections of the bundle V.

Suppose therefore from now on that V is a line bundle, with curvature form Θ. Assume also for the moment that the Kähler form ω of M is proportional to Θ: $i\Theta(x) = \lambda(x)\omega(x)$, where λ is a smooth function. If M is a Riemann surface this is automatic, of course; otherwise the metric may have to be suitably massaged—this will be discussed later. Let Λ denote the operator of interior multiplication by the Kähler form ω, L the operator of exterior multiplication by ω. Write the connection on V as $\nabla + \bar\partial$, where ∇ is of type $(1,0)$; and recall [12] the Kähler identities

$$[\Lambda,\bar\partial] = -\tfrac{1}{2}i\nabla^*, \qquad [\Lambda,\nabla] = -\tfrac{1}{2}i\bar\partial^*,$$
$$[\Lambda, L] = (n-p-q) \quad \text{on } (p,q)\text{-forms.}$$

Now

$$\Theta = \bar\partial\nabla + \nabla\bar\partial = \{\bar\partial,\nabla\},$$

where the curly brackets denote the anticommutator. Hence from the Jacobi identity

$$[\Lambda,\Theta] = [\Lambda,\{\bar\partial,\nabla\}] = \{[\Lambda,\bar\partial],\nabla\} + \{[\Lambda,\nabla],\bar\partial\}$$
$$= -\tfrac{1}{2}i\{\nabla^*,\nabla\} + \tfrac{1}{2}i\{\bar\partial^*,\bar\partial\}.$$

But also

$$[\Lambda,\Theta] = -i\lambda[\Lambda,L] = -i\lambda(n-p-q).$$

Hence

(4.5) $$\Delta_{p,q} = \nabla^*\nabla + \nabla\nabla^* - 2\lambda(n-p-q).$$

This is the Weitzenbock formula in the present context.

Let us call V *uniformly positive* if the function λ is uniformly positive in the sense of (1.13), and *uniformly negative* if $-\lambda$ is uniformly positive.

(4.6) Proposition. *With the assumptions above, let $c(V)$ denote the Chern class of V in $H^2_\beta(M)$. Then*:

(a) *If V is uniformly positive then* $h_{n,0} = h'_{n,0} = (-1)^n\langle m, e^{-c(V)} \cup Td(TM)\rangle$;

(b) *If V is uniformly negative then* $h_{0,n} = h'_{0,n} = (-1)^n\langle m, e^{c(V)} \cup Td(TM)\rangle$.

Proof. The two statements are equivalent by Kodaira-Serre duality. We prove (b). Write $\Delta_+ = \Delta_{0,n} \oplus \Delta_{0,n-2} \oplus \cdots$, $\Delta_- = \Delta_{0,n-1} \oplus \Delta_{0,n-3} \oplus \cdots$. Then the Weitzenbock formula (3.4) yields $(\Delta_-\alpha, \alpha) \geqslant (\lambda\alpha, \alpha)$. By the principle of convergence transfer (1.13), then, $e^{-t\Delta_-}$ and $e^{-t\Delta_+}$ converge fully, so $h_{0,p} = h'_{0,p}$ for all p. But (from (4.5) again) a harmonic $(0, p)$-section for $p < n$ is necessarily zero; thus $h_{0,p} = 0$ for $p < n$. The result follows from (4.4).

We must now analyze the assumption that Θ is a multiple of the Kähler form. In the usual proof [12] of the Kodaira vanishing theorem one uses the curvature Θ to define a new Kähler metric. This can also be done in the noncompact situation.

(4.7) Theorem. *Let M be a Kähler manifold of bounded geometry equipped with a regular exhaustion and corresponding fundamental class m. Let V be a holomorphic line-bundle over M. Then*

(a) *If V is uniformly positive then $h_{n,0} = h'_{n,0} = (-1)^n\langle m, e^{-c(V)} \cup Td(TM)\rangle$;*

(b) *If V is uniformly negative then $h_{0,n} = h'_{0,n} = (-1)^n\langle m, e^{c(V)} \cup Td(TM)\rangle$.*

Proof. Assume V is uniformly positive. The result will follow from (4.6) once it is shown that $i\Theta$ is the Kähler form of a metric of bounded geometry on M strictly quasi-isometric to the old one. Since Θ is bounded below, the two metrics are quasi-isometric in the weaker sense, and hence the new metric has positive injectivity radius. Since Θ is C^∞-bounded, the new metric has bounded geometry. Moreover, its connection coefficients are uniformly bounded in the old metric. Thus the old and new metrics are strictly quasi-isometric.

(4.8) Corollary. *Let V be a uniformly positive line bundle over a manifold M as above. Then some tensor power of V admits an infinite-dimensional space of L^2 holomorphic sections.*

Proof. Just imitate the proof of the corresponding result for compact manifolds (see [12, Chapter II]).

Under what circumstances can one construct a uniformly positive metric on V? For compact manifolds one knows that positivity is a topological property of line bundles. Here one can modify the metric on certain line bundles so as to make it positive. To do this we shall need to make use of diffusion acting on UC^∞ forms.

(4.9) Lemma. *Let Δ be the total Laplacian acting on forms on M. Let $k_t(x, y)$ be the Schwartz kernel of $e^{-t\Delta}$. Then*

(a) *$|k_t(x, \cdot)|$ is integrable for each x;*

(b) *If η is a UC^∞ form then so is $e^{-t\Delta}\eta$ defined by*

$$e^{-t\Delta}\eta = \int_y k_t(\cdot, y)\eta(y);$$

(c) *The PDE*

$$(\partial/\partial t)e^{-t\Delta}\eta = -\Delta e^{-t\Delta}\eta$$

is satisfied for all UC^∞ forms η.

Proof. It is enough to establish that for each multi-index α and fixed t,

$$\int |\nabla_x^\alpha k_t(x, y)|\text{vol}(y) < C_\alpha < \infty$$

with C_α independent of x. Since $e^{-t\Delta}$ is a uniform operator of order $-\infty$ we know [19, 5.4] that

$$\int_{d(x,y)>R} |k_t(x, y)|^2 \cdot \text{vol}(y) \leqslant \mu(R)^2,$$

where $\mu(R) \to 0$ as $R \to \infty$, and in fact the explicit calculation of [19, 5.6] gives

$$\mu(R) \leqslant t^{-1/2}\int_{\lambda>R} P(\lambda)e^{-\lambda^2/4t}\,d\lambda \quad (P \text{ being a polynomial in } \lambda),$$
$$\leqslant Ae^{-aR^2}$$

for some constants A and a. On the other hand one knows from the Sobolev estimates that $\nabla_x^\alpha k_t(x, y)$ has a uniform bound independent of x and y. An interpolation argument therefore yields

$$|\nabla_x^\alpha k_t(x, y)| \leqslant B_\alpha e^{-bR^2}$$

where $R = \text{dist}(x, y)$. However the volume comparison theorem [19, 2.3] gives

$$\text{vol}\, B(x, R) \leqslant Ce^{\mu R}$$

for some constants C and μ. So we may write

$$\begin{aligned}\int |\nabla_x^\alpha k_t(x, y)| &\leqslant \int_{R=0}^\infty B_\alpha e^{-bR^2}\,d\,\text{Vol}(B(x, r)) \\ &= \int_0^\infty 2B_\alpha Rbe^{-bR^2}\,\text{Vol}(B(x, r))\,dR \\ &\leqslant B_\alpha Cb\int_0^\infty Re^{-bR^2+\mu R}\,dR = C_\alpha.\end{aligned}$$

This proves the result.

Let V be a holomorphic vector-bundle over the hermitian manifold M. We will say that two hermitian metrics on V are *strictly quasi-isometric* if each is dominated by a constant multiple of the other and their hermitian connections are boundedly equivalent in the sense of [19, 3.7].

(4.10) Proposition. *Let Θ be the curvature form of a metric on the line-bundle V over the Riemann surface M of bounded geometry. Then for any $t \geqslant 0$ there is a metric on V strictly quasi-isometric to the original one with curvature $e^{-t\Delta}\Theta$.*

Proof. We have,

$$\Theta - e^{-t\Delta}\Theta = \int_0^t (\Delta e^{-s\Delta}\Theta)\, ds = \Delta\beta, \quad \text{where } \beta = \int_0^t e^{-s\Delta}\Theta\, ds.$$

However, on a Riemann surface $\Delta\beta = \partial\bar{\partial}\rho$, where $\rho = 4(*\beta)$. Clearly ρ is C^∞-bounded. The metric on V given by multiplication of the original metric by e^ρ is strictly quasi-isometric to the original one, and has curvature form $\Theta - \partial\bar{\partial}\rho = e^{-t\Delta}\Theta$ (cf. [12, p. 149]).

One may regard $e^{-t\Delta}$ as a "moving average" operator. Thus the content of the proposition is that if a suitable "moving average" of Θ is uniformly positive, then the bundle can be remetrized so that Θ itself is uniformly positive.

This theory will now be applied to the Riemann-Roch problem in the plane. Let Γ be a discrete subset of $\mathbf{C}$. It will be called a *pseudo-lattice* if there is a number $r > 0$ such that each ball of radius r contains at most one point of Γ. A *divisor* D on $\mathbf{C}$ will be a finite formal linear combination of pseudo-lattices with integer coefficients, and a meromorphic function f on $\mathbf{C}$ will be called *subordinate* to D if for all z in $\mathbf{C}$ the order of the pole of f at z is no greater than the order to which z appears in D. (Here we count zeros as poles of negative order.)

The classical theorems of Weierstrass and Mittag-Leffler (cf. [13]) assure us that there exist many meromorphic functions subordinate to any divisor D. To obtain a Riemann-Roch type result one must impose growth conditions. Define the function $\mu(z) = \min(l, |z|)$ for some positive l smaller than half the minimum distance between points of Γ. For a pseudo-lattice Γ let

$$\mu_\Gamma(z) = \prod_{w\in\Gamma} \mu(z - w),$$

and for a divisor $D = \Sigma n_i\Gamma_i$ let $\mu_D(z) = \Pi(\mu_{\Gamma_i}(z))^{n_i}$. A function f will be called *L^2-subordinate* to D if it is subordinate to D and $\int \mu_D^2|f|^2 < \infty$. It is easy to see that this condition is independent of the choice of l. We ask: When do there exist meromorphic functions L^2-subordinate to D?

The first result on this question was obtained by Alain Connes as a corollary of the foliation index theorem.

Theorem [6]. *Suppose that Γ_1 and Γ_2 are lattices in $\mathbf{C}$ with* density(Γ_1) > density(Γ_2). *Then for almost all translations $z \in \mathbf{C}$ there is an infinite-dimensional space of meromorphic functions L^2-subordinate to $[z + \Gamma_1] - [\Gamma_2]$.*

Connes' work left open the question of whether the restriction to almost all z was necessary. It will emerge from our analysis that it was not.

To relate the question to index theory one can construct a line bundle V_D from the divisor D in the usual sort of way. Explicitly, let E be a holomorphic line-bundle over **C** with an hermitian metric, and let p be a point of **C**. Say for definiteness that the radius l is equal to 5, and suppose that, over the ball of radius 4 around p, E is isometric to the trivial bundle. A new bundle F, said to be *obtained from E by patching in a pole at p*, is defined as follows. On the complement U of $B'(p, 2)$, F is isometric to E; on the ball $V = B(p, 3)$, F is trivial; the transition function is $(z - p)$ (i.e., if s_U and s_V are local representations of a section s on U and V, then $s_V = (z - p)s_U$). As for the metric on $F_{|V}$: if s is the standard section over V, then we are forced to take $|s(z)|^2 = |z - p|^2$ for $2 \leqslant |z - p| \leqslant 3$; we set $|s(z)| = \phi(|z - p|)$ for $|z - p| \leqslant 3$, where ϕ is a smooth function fixed once and for all, constant in a neighborhood of zero, and such that $\phi(r) = r$ for $2 \leqslant r \leqslant 3$.

Notice the following points with regard to this construction:

(i) A holomorphic L^2 section of F may be identified with a meromorphic section s of E such that $\int |s|^2 w < \infty$, where $w(z) = |z - p|^2$ for $|z - p| \leqslant 1$, $w(z) = 1$ for $|z - p| > 1$. Indeed, given a section (s_U, s_V) of F, one sees that $(s_U, s_V/(z - p))$ describes a section of E. For the original section to be square integrable it is necessary and sufficient that both s_U and s_V be square integrable, so the result is immediate.

(ii) One has

$$\int_{|z-p|<3} c(z) = 1,$$

where c is the Chern form of F, $c = i\Theta/2\pi$. Indeed

$$\int_{|z-p|<3} c(z) = \int_{|z-p|<3} (-i/2\pi)\partial\bar{\partial}\log|\phi|^2$$

$$= (1/4\pi)\int (\partial/\partial n)\log|\phi|^2\, ds \quad \text{(Green's theorem)}$$

$$= (1/4\pi)\cdot(2/r)\cdot 2\pi r = 1.$$

Dually to this construction, one can *patch in a zero at p*. Now the line-bundle V_Γ associated to a pseudo-lattice Γ is obtained from the trivial bundle by patching in zeros and poles with appropriate multiplicities at the points of Γ. We define the line-bundle V_D associated to a divisor D in the usual way as the tensor product of the line-bundles associated to its constituent pseudo-lattices. Clearly, V_D has bounded geometry. From (i) above, an L^2 holomorphic section of V_D may be identified with a meromorphic function L^2-subordinate to D. From (ii), if m is a fundamental class associated to a regular exhaustion and $D = [\Gamma_1] - [\Gamma_2]$, then

$$\langle c(V_D), m\rangle = \text{Density}\,\Gamma_1 - \text{Density}\,\Gamma_2$$

(assuming that the r.h.s. exists), the density being measured with respect to the given regular exhaustion.

The Riemann-Roch theorem for a line-bundle V over $\mathbf{C}$ takes the simple form

$$h'_{0,0}(V) - h'_{0,1}(V) = \langle c(V), m\rangle.$$

If (0.1) held, so that $h_{0,0}(V) = h'_{0,0}(V)$, then Connes' result (without the "almost everywhere" condition) would be an immediate corollary; for if $h_{0,0}(V_D) > 0$, then there is an infinite-dimensional space of meromorphic functions L^2-subordinate to D. However, we will now present an example to show that (0.1) does not hold in general.

Choose any lattice Γ in $\mathbf{C}$ that is symmetrical about the origin and does not meet the imaginary axis. Let Γ_1 be the intersection of Γ with the left-hand half-plane, Γ_2 be the intersection of Γ with the right-hand half-plane, and let D be the divisor $[\Gamma_1] - [\Gamma_2]$.

(4.11) Lemma. *There are no nonzero meromorphic functions L^2-subordinate to D.*

Proof. Let f be such a function and let $g(z) = f(z)f(-z)$. By the symmetry of Γ, $f(-z)$ has a zero wherever $f(z)$ has a pole, so g is a holomorphic function on $\mathbf{C}$. Moreover, the weight function μ_D satisfies $\mu_D(z)\mu_D(-z) = 1$. It follows that g is integrable. Hence g is identically zero.

So $h_{0,0}(V_D) = 0$ for any exhaustion. On the other hand, it is easy to construct a regular exhaustion of $\mathbf{C}$ so skewed that $\langle m, c(V_D)\rangle > 0$, so that $h'_{0,0}(V_D) > 0$. We conclude that (0.1) does not hold for the $\bar\partial$ operator on V_D.

Intuitively, the problem arises because the sets Γ_1 and Γ_2 are far apart, rather than mixed up together as in Connes' example. The following notion formalizes the idea that "Density $\Gamma_1 >$ Density Γ_2 uniformly over $\mathbf{C}$."

(4.12) Definition. We say $D = [\Gamma_1] - [\Gamma_2]$ is *uniformly positive* if there exist $\eta > 0$ and $r_0 > 0$ such that for all $x \in \mathbf{C}$ and all $r > r_0$,

$$(1/\pi r^2)(\#(\Gamma_1 \cap B(x,r)) - \#(\Gamma_2 \cap B(x,r))) > \eta.$$

(4.13) Lemma. *If D is uniformly positive, then there is a uniformly positive metric in the quasi-isometry class on V_D.*

Proof. Let A_r denote the averaging operator

$$A_r f = -\frac{1}{2i\pi r^2}\int_{|z|<r} f(z)\,dz \wedge d\bar z.$$

Write $c(V_D) = \lambda\omega$, where ω is the standard Kähler form. It is clear from Definition 4.12 that for all sufficiently large r, $A_r\lambda > \eta/2$.

From (4.10) one sees that it is enough to prove that $e^{-t\Delta}\lambda > \varepsilon > 0$ for some large t. However, one may write

$$e^{-t\Delta}\lambda(0) = \frac{1}{4\pi t}\int\int e^{-r^2/4t}\lambda(re^{i\theta})\,r\,dr\,d\theta$$
$$= \frac{1}{8t^2}\int r^3 e^{-r^2/4t}A_r\lambda\,dr$$

on integration by parts. It is therefore clear that $e^{-t\Delta}\lambda(0) > \eta/4$ for sufficiently large t, and the desired result follows from this since all the estimates are translation-invariant.

Combining (4.13) and (4.6) we obtain

(4.14) Theorem (*planar Riemann-Roch*). *Let* Γ_1 *and* Γ_2 *be pseudo-lattices in* $\mathbf{C}$. *If* $D = [\Gamma_1] - [\Gamma_2]$ *is uniformly positive, then* $h_{0,0}(V_D) =$ Density $\Gamma_1 -$ Density $\Gamma_2 > 0$ (*in any regular exhaustion*). *Consequently, there is an infinite-dimensional space of meromorphic functions* L^2*-subordinate to* D.

In particular, the "almost everywhere" condition can be dropped from the statement of Connes' result. The possibility of this extension of Connes' theorem was suggested to me by Atiyah.

We close with a couple of questions.

(a) The result shows that if Γ is uniformly positive then there is an infinite-dimensional space of meromorphic functions L^2-subordinate to Γ. How far can the hypothesis of uniform positivity be weakened? In particular, if Γ is contained in a half-plane and uniformly positive there, does the conclusion still hold?

(b) Can any direct significance be attached to the numerical value (and not merely the positivity) of $h_{0,0}(V)$? One would expect that if $h_{0,0}(V_D) > h_{0,0}(V_{D'})$, then there should be "more" meromorphic functions L^2-subordinate to D than to D'. Can this be made precise?

5. Relation with other L^2-index theorems

Atiyah in [2] and Connes in [6] have described L^2 index theorems on noncompact manifolds, which overlap with the one given here. In this section the relationship between these various results will be discussed. This will enable us to clarify the significance of the convergence problem which has formed the subject of the present paper, and which does not arise in the other two works.

First we consider the theorem of Atiyah. This applies to a manifold $\tilde{X}$ which is a Galois covering with Galois group Γ of a compact manifold X, and an operator $\tilde{D}$ which is the lifting to $\tilde{X}$ of an operator D on X. Let P^+ and P^- denote the orthogonal projections (in L^2) onto the kernel and cokernel of $\tilde{D}$, with Schwartz kernels k^+ and k^-, and let F be a fundamental domain for Γ. Then the Atiyah theorem says

$$\int_F (\operatorname{Tr} k^+ - \operatorname{Tr} k^-) = \operatorname{Index} D = \int_F \mathbf{I}(\tilde{D}). \tag{5.1}$$

To relate this to our theorem, suppose that Γ is amenable. This ensures that $\tilde{X}$ is regularly exhaustible [19, 6.6] (in fact the converse implication also holds; see [5]). If m and τ are associated to the regular exhaustion, then (5.1) becomes

$$\tau(P^+) - \tau(P^-) = \langle m, \mathbf{I}(\tilde{D})\rangle.$$

This is (0.1). Thus this particular case of Atiyah's theorem can be obtained from ours provided we prove the full convergence of e^{-tD^2}. In fact the full convergence can be obtained in a more general case:

(5.2) Lemma. *Suppose that M is a regularly exhaustible noncompact manifold of bounded geometry equipped with a Dirac-type operator D. If there is a group G of diffeomorphisms of M preserving all the structures and such that M/G is compact, then* (0.1) *holds.*

(Notice that G need not be discrete, nor M/G a manifold.)

Proof. Let k_t^+ be the Schwartz kernel of $e^{-tD^2}(1+\eta)/2$. We want to show that $\operatorname{Tr} k_t^+ \to \operatorname{Tr} k^+$ uniformly on M. Certainly $\operatorname{Tr} k_t^+$ decreases to $\operatorname{Tr} k^+$ by monotoneity. But $\operatorname{Tr} k_t^+$ and $\operatorname{Tr} k^+$ are G-invariant, so descent to the compact metric space M/G. By Dini's theorem, the convergence is uniform.

The general case of Atiyah's theorem (where Γ is nonamenable) does not follow from ours. Atiyah constructs a trace using the extra data provided by the Γ-equivariant structure, whereas our method takes no account of this. It is however possible to use our heat equation estimates to prove Atiyah's theorem in general. This is essentially the proof outlined in §6.2 of [2].

We turn now to Connes' theorem. Here the noncompact manifolds are leaves of a foliation $\mathscr{F}$ of some compact manifold X. It is supposed that $(X, \mathscr{F})$ has an invariant transverse measure Λ. Let D be a differential operator on X which restricts to leaves and is elliptic along them. Define functions k^+ and k^- on X as follows: $k^+(x)$ (resp. $k^-(x)$) is the trace at (x, x) of the Schwartz kernel of the projection onto the kernel of D (resp. D^*) in the L^2-space of the holonomy covering of the leaf through x. If $[\Lambda]$ is the Ruelle-Sullivan homology class corresponding to Λ, then Connes' theorem says that

$$\int (k^+ - k^-)\, d\Lambda = \langle \mathbf{I}(D), [\Lambda]\rangle. \tag{5.3}$$

Suppose that $\mathscr{F}$ has a regularly exhaustible leaf L. The Plante construction [16] yields a transverse measure Λ on $\mathscr{F}$ with the property that

$$\int f\,d\Lambda = \langle m, f_{|L}\rangle$$

for all continuous functions f on X, where m is a fundamental class on L associated with the regular exhaustion. One might then think that (5.3) should imply (0.1) for any pair (L, D) that can be embedded as a leaf in a foliation of a compact manifold. However this is not the case: the functions k^+ and k^- need not be continuous. In fact, since L is likely to have Λ-measure zero, Connes' theorem in this form does not immediately allow us to make statements about an individual leaf. This phenomenon is responsible for the appearance of "almost all translations" in Connes' version of the Riemann-Roch theorem on the plane.

Thus there are situations in which our theorem can give more precise information than Connes'. On the other hand, the range of applicability of Connes' theorem is wider. There are plenty of foliations with transverse measures but without regularly exhaustible leaves—one example is discussed in [6]—and the K-theoretic version of the foliation index theorem due to Connes and Skandalis [8] dispenses with the transverse measure altogether.

Further discussion of the relation of our work with Connes' will appear in [18].

6. Further remarks

We conclude with some miscellaneous comments and questions.

(6.1) Extravagance in derivatives. We made no effort to use the minimum possible number of derivatives to get the required estimates. For example, our definition of "bounded geometry" requires uniform bounds on the curvature and on all its covariant derivatives. Can we get away with just uniformly bounded curvature? A related question: Under what circumstances can strict quasi-isometry be replaced by ordinary quasi-isometry?

(6.2) Betti numbers. As was noted in §2, it is an outstanding question whether the two sorts of Betti numbers defined there are equal, $\beta = \beta'$. If not, which is the "good" definition? The index-theorem suggests β'; and I hope to show elsewhere that the numbers β' also satisfy "Morse inequalities" with respect to a suitably well-behaved Morse function. This uses the methods of Witten [21]. Unfortunately, even if the β' are the "good" Betti numbers, the β are the interesting ones.

It is also possible to use the methods of classification theory [20] to prove $\beta = \beta'$ on Riemann surfaces satisfying strong parabolicity conditions.

(6.3) Metrics on nonamenable coverings. The statements of Propositions 2.9 and 3.3 make sense (if one replaces "large" by "cocompact") without the assumption that the covering group Γ is amenable. However the proof completely breaks down: our index theorem is inapplicable since there is no regular exhaustion, and Atiyah's is inapplicable since the Γ-equivariance is destroyed by perturbations of the metric. Do these results still hold?

A possible approach to this question would be to prove:

(a) The K-theoretic index of D is quasi-isometry invariant (in a suitable sense).

(b) The natural map

$$K(S(X) \rtimes \Gamma) \to K(\mathscr{U}_{-\infty}(\tilde{X})),$$

where $S(X)$ denotes the algebra of smoothing operators on X, is injective.

This latter statement says that we lose no information at the K-theory level by forgetting the equivariance. It may be related to the next question.

(6.4) Application of cyclic cohomology. The fundamental class on M gives a trace on $\mathscr{U}_{-\infty}(M)$, which detects certain elements of $K(\mathscr{U}_{-\infty}(M))$. Can one construct, even if M is not regularly exhaustible, higher cyclic cocycles on $\mathscr{U}_{-\infty}(M)$ which detect other data in $K(\mathscr{U}_{-\infty}(M))$?

(6.5) Relation to value-distribution theory. The ideas of "regular exhaustion" and "mean Euler characteristic" were introduced by Ahlfors [1] in his discussion of covering surfaces and value-distribution theory. Can these be related to index theory (as advertised in [3])? Specifically, can Ahlfors' "Metrisch-topologischer Hauptsatz" be deduced from the L^2 Gauss-Bonnet or Riemann-Roch theorems given here? (The assumption of bounded geometry seems to be a problem.) J. Dodziuk [10] has noted the relationship between Ahlfors' results and questions in L^2 index theory.

(6.6) Low energy physics. It is interesting to notice that the convergence problems discussed in this paper all involve low energies and large distances, in contrast with the more familiar "high-energy" calculus of pseudo-differential operators. In this context, S. Hurder has made the interesting suggestion that the difference between the left- and right-hand sides of (0.1) should be regarded as a "low energy η-invariant." Does this "invariant" depend only on the asymptotic behavior of M? Does it vanish in the Gauss-Bonnet case?

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